A Rewrite of the Lane-Emden Equation using Relative Area

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1 Introduction

The Lane-Emden (LE) equation models a spherically symmetric polytropic gas cloud with polytropic index n. It can be written using a variable x denoting the relative radius from the center of the cloud, $0 < x < 1$. The LE equation is

$$\frac{1}{x^2} \frac{d}{dx} \left[x^2 \frac{df}{dx} \right] + f^n = 0 \tag{1}$$

where f is a function of x to be found. The boundary conditions of this second order differential equation are $f(0) = 1$ and $df/dx(0) = 0$. Given a knowledge of f (which is unitless) and the polytropic index n, the relative density profile of the cloud can be calculated.

A couple of references for the LE equation are the textbook [FL10], section 5.4, and the classic book by Chandrasekhar, [SC39], chapter 4. For power series expansions, the coefficients written as rational fractions involving the polytropic index n, up to the degree x^{28} term, see Rohe's paper [KR15].

As Chandrasekhar notes, f is an even function, so for instance all odd-degree terms in its power series expansion about $x = 0$ have zero coefficients. That is,

$$f(x) = 1 - \frac{1}{6} x^2 + \frac{n}{120} x^4 + ... \tag{2}$$

with successive terms x^6, x^8, etc as given in Rohe's paper.

That representation of f raises the question, whether it might be useful to rewrite the LE equation with a different independent variable, let's call it $s = x^2$. Here s is a proxy for the relative surface area of a spherical shell of relative radius x. Since $0 < x < 1$, we have $0 < s < 1$. Thus s represents the relative surface area, compared to the surface area of the outside boundary of the gas cloud.

A representation in terms of relative surface area may be physically suggestive. The physics which is being modeled by the LE equation, involves transport of energy and force between nested spherical shells. Thus a look at the graphs of f vs independent variable s, and of the derivative df/ds, may help with interpretation of a polytropic gas cloud model.

2 Rewrite of LE Equation

In this section, the LE equation (1) in terms of relative radius x, will be rewritten to obtain an equation in terms of relative surface area s. The result will be

$$6\,\frac{df}{ds} + 4s\,\frac{d^2 f}{ds^2} + f^n = 0 \tag{3}$$

The details of the rewrite follow.

For any function g, taking the derivative with respect to an independent variable x or an independent variable $s = x^2$, we have

$$\frac{dg}{dx} = \frac{dg}{ds}\,\frac{ds}{dx} = \frac{dg}{ds}\,\frac{d}{dx}\left(x^2\right) = 2x\,\frac{dg}{ds}$$

Hence the LE equation (1) becomes

$$\frac{2x}{x^2}\,\frac{d}{ds}\left[x^2\,\frac{df}{dx}\right] + f^n = 0$$

or

$$\frac{2}{x}\,\frac{d}{ds}\left[x^2\,2x\,\frac{df}{ds}\right] + f^n = 0$$

or

$$\frac{2}{s^{1/2}}\,\frac{d}{ds}\left[2s^{3/2}\,\frac{df}{ds}\right] + f^n = 0$$

Calculating out that last equation, gives

$$\frac{2}{s^{1/2}}\left[3s^{1/2}\,\frac{df}{ds} + 2s^{3/2}\,\frac{d^2 f}{ds^2}\right] + f^n = 0$$

That in turn simplifies to

$$6\,\frac{df}{ds} + 4s\,\frac{d^2 f}{ds^2} + f^n = 0$$

which is equation (3) as claimed.

References

[FL10] Francis LeBlanc, *An Introduction to Stellar Astrophysics*, Wiley, 2010.

[SC39] S. Chandrasekhar, *An Introduction to the Study of Stellar Structure*, Chicago, 1939. Dover reprint, 1958.

[KR15] Klaus Rohe, "Computing the coefficients for the power series solution of the Lane-Emden equation with the Python library SymPy", Arxiv:1409.2008, revision v2, 25-Jan-2015.